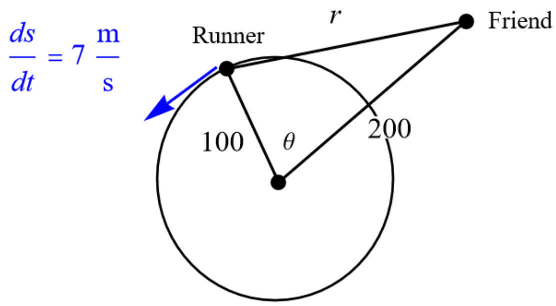
Exercise 49

A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?

Solution

Draw a schematic of the track at a certain time.



The aim is to find dr/dt when $r = 200$. Start with the formula relating the sides of this triangle, the law of cosines.

$$\begin{aligned} r^2 &= 100^2 + 200^2 - 2(100)(200)\cos\theta \\ &= 50\,000 - 40\,000\cos\theta \end{aligned} \tag{1}$$

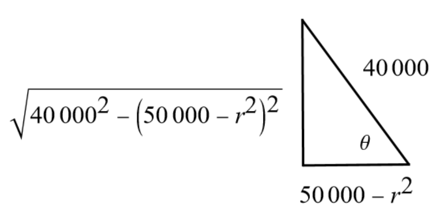
Take the derivative of both sides with respect to time by using the chain rule.

$$\begin{aligned} \frac{d}{dt}(r^2) &= \frac{d}{dt}(50\,000 - 40\,000\cos\theta) \\ 2r \cdot \frac{dr}{dt} &= -40\,000(-\sin\theta)\cdot\frac{d\theta}{dt} \\ r\frac{dr}{dt} &= 20\,000(\sin\theta)\frac{d\theta}{dt} \end{aligned} \tag{2}$$

In order to write $\sin\theta$ in terms of r, solve equation (1) for $\cos\theta$,

$$\cos\theta = \frac{50\,000 - r^2}{40\,000},$$

and draw the implied right triangle.



As a result,

$$\sin\theta = \frac{\sqrt{40\,000^2 - (50\,000 - r^2)^2}}{40\,000},$$

which means equation (2) becomes

$$\begin{aligned} r\frac{dr}{dt} &= 20\,000(\sin\theta)\frac{d\theta}{dt} \\ &= 20\,000\left[\frac{\sqrt{40\,000^2 - (50\,000 - r^2)^2}}{40\,000}\right]\frac{d\theta}{dt} \\ &= \frac{\sqrt{40\,000^2 - (50\,000 - r^2)^2}}{2}\frac{d\theta}{dt}. \end{aligned}$$

Solve for dr/dt.

$$\frac{dr}{dt} = \frac{\sqrt{40\,000^2 - (50\,000 - r^2)^2}}{2r}\frac{d\theta}{dt}$$

Note the formula for arclength on this track: $s = 100\theta$. Take the derivative of both sides with respect to t: $\frac{ds}{dt} = 100\frac{d\theta}{dt}$. This means that

$$\begin{aligned} \frac{dr}{dt} &= \frac{\sqrt{40\,000^2 - (50\,000 - r^2)^2}}{2r}\left(\frac{1}{100}\frac{ds}{dt}\right) \\ &= \frac{\sqrt{40\,000^2 - (50\,000 - r^2)^2}}{2r}\left(\frac{1}{100}\cdot 7\right) \\ &= \frac{7\sqrt{40\,000^2 - (50\,000 - r^2)^2}}{200r}. \end{aligned}$$

Therefore, when the distance between the friend and runner is 200 meters, the rate of change of this distance with respect to time is

$$\left.\frac{dr}{dt}\right|_{r=200} = \frac{7\sqrt{40\,000^2 - [50\,000 - (200)^2]^2}}{200(200)} = \frac{7\sqrt{15}}{4}\ \frac{\text{m}}{\text{s}} \approx 6.77772\ \frac{\text{m}}{\text{s}}.$$